P6.

$$\int_0^1 \phi_i(-p\phi''+q\phi-f)\,dx$$

$$= -\int_0^1 \phi_i p\phi''\,dx + \int_0^1 q\phi_i\phi\,dx - \int_0^1 f\phi_i\,dx$$

$$= -\phi_i p\phi'\Big|_0^1 + \int_0^1 \phi'\frac{d}{dx}(p\phi_i)\,dx + \int_0^1 q\phi_i\phi\,dx - \int_0^1 f\phi_i\,dx$$

$$= \int_0^1 p\phi'\phi_i'\,dx + \int_0^1 p'\phi_i\phi'\,dx + \int_0^1 q\phi_i\phi\,dx - \int_0^1 f\phi_i\,dx$$

$$= \sum_j C_j \int_0^1 p\phi_j'\phi_i'\,dx + \sum_j C_j \int_0^1 p'\phi_i\phi_j'\,dx + \sum_j \int_0^1 C_j q\phi_i\phi_j\,dx - \int_0^1 f\phi_i\,dx$$

$$\sum_j C_j\left[\int_0^1 (p\phi_i'\phi_j' + p'\phi_i\phi_j' + q\phi_i\phi_j)\,dx\right] = \int_0^1 f\phi_i\,dx$$

若：p与q为常量：

$$\sum_j C_j\left[\int_0^1 (p\phi_i'\phi_j' + q\phi_i\phi_j)\,dx\right] = \int_0^1 f\phi_i\,dx$$

$\phi_i' = \frac{1}{h_i}$, $x\in[x_{i-1},x_i]$; $\frac{1}{h_{i+1}}$, $x\in[x_i,x_{i+1}]$; 0, otherwise.

≠0，当且仅当 $|i-j|\le 1$.

(1) $i=j$: $\int_0^1 (p\phi_i'^2 + q\phi_i^2)\,dx = p\cdot\left(\frac{1}{h_i^2}\cdot h_i + \frac{1}{h_{i+1}^2}h_{i+1}\right)$

$$+ q\int_{x_{j-1}}^{x_j}\left(\frac{x-x_{j-1}}{x_j-x_{j-1}}\right)^2 dx + q\int_{x_j}^{x_{j+1}}\left(\frac{x_{j+1}-x}{x_{j+1}-x_j}\right)^2 dx$$

$$= p\left(\frac{1}{h_i}+\frac{1}{h_{j+1}}\right) + q\cdot\frac{1}{3}(h_j+h_{j+1})$$

(2) $i=j-1$, $C_j\left[\int_{x_i}^{x_{i+1}} p\cdot\frac{-1}{h_{i+1}}\cdot\frac{1}{h_{i+1}} + \int_{x_i}^{x_{i+1}} q\cdot\frac{x-x_i}{x_{i+1}-x_i}\cdot\frac{x_{i+1}-x}{x_{i+1}-x_i}\,dx\right]$

$$= -p\cdot\frac{1}{h_{i+1}}\cdot + (-q)\cdot\frac{1}{h_{i+1}^2}\left(\frac{1}{3}x_{i+1}^3-\frac{1}{3}x_i^3-(x_i+x_{i+1})\frac{1}{2}(x_{i+1}^2-x_i^2)+x_ix_{i+1}(x_{i+1}-x_i)\right)$$

$$= -\frac{p}{h_{i+1}}-\frac{q}{h_{i+1}}\left(\frac{1}{3}x_i^2+\frac{1}{3}x_{i+1}^2+\frac{x_ix_{i+1}}{3}-\frac{1}{2}(x_i^2+x_{i+1}^2+2x_ix_{i+1})+x_ix_{i+1}\right)$$

$$= -\frac{p}{h_{i+1}}+\frac{q}{h_{i+1}}\left(\frac{1}{6}x_i^2+\frac{1}{6}x_{i+1}^2 - \frac{1}{3}x_ix_{i+1}\right)$$

$$= -\frac{p}{h_{i+1}}+\frac{1}{6}qh_{i+1}$$

(3) $j=i-1$, $-\frac{p}{h_i}+\frac{1}{6}qh_i$

$$\begin{pmatrix} p\left(\frac{1}{h_1}+\frac{1}{h_2}\right) & -\frac{p}{h_2}+\frac{1}{6}qh_2 & 0 & \cdots & \\ -\frac{p}{h_2}+\frac{1}{6}qh_2 & \frac{p}{h_2}+\frac{p}{h_3} & -\frac{p}{h_3} & & \\ & -\frac{p}{h_3} & \ddots & & -\frac{p}{h_{N-1}} \\ & & & \ddots & \\ & & & -\frac{p}{h_{N-1}} & \frac{p}{h_{N-1}}+\frac{p}{h_N} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{N-1}\end{pmatrix}$$

$$+\begin{pmatrix} \frac{q}{3}(h_1+h_2) & \frac{1}{6}qh_2 & & & \\ \frac{1}{6}qh_2 & \frac{q}{3}(h_2+h_3) & & & \\ & & \ddots & & \\ & & & \ddots & \frac{1}{6}qh_{N-1} \\ & & & \frac{1}{6}qh_{N-1} & \frac{q}{3}(h_{N-1}+h_N) \end{pmatrix}\begin{pmatrix} c_1 \\ \vdots \\ \vdots \\ c_{N-1}\end{pmatrix}$$

$$= \begin{pmatrix} \int f\phi_1\,dx \\ \int f\phi_2\,dx \\ \vdots \\ \int f\phi_{N-1}\,dx \end{pmatrix}$$

若有 ϕ_0 与 ϕ_N，则

$$\phi(x)=\sum_{j=0}^{N} c_j\phi_j(x)$$

$$c_0=\phi(0),\ c_N=\phi(1)$$

则额外会有

(1) $j=0$, $i=1$. $\int_0^1 \phi_1'(x)\phi_0'(x)p(x)dx+\int_0^1 q\phi_1\phi_0 dx$

$$=-\frac{p}{h_1}+\frac{1}{6}qh_1$$

$$\int f\phi_1 dx\to\int f\phi_1 dx+\frac{p}{h_1}c_0-\frac{1}{6}qh_1\cdot c_0$$

(2) $j=N$, $i=N-1$.

$$\int_0^1 p\phi_{N-1}'\phi_N'+q\phi_{N-1}\phi_N dx$$

$$=-\frac{p}{h_N}+\frac{1}{6}qh_N$$

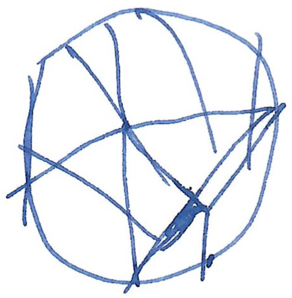
$$\Rightarrow\int f\phi_{N-1}dx\to\int f\phi_{N-1}dx+\frac{p}{h_N}c_N-\frac{1}{6}qh_Nc_N$$

$\int f\phi_i\,dx$：(1) f：常数：在 $[x_i,x_{i+1}]$ 中 $f=\frac{1}{2}(f_i+f_{i+1})$
(2) f：线性函数.

2D：试探函数仍为局域线性函数.

1D需线段，2D需三角形，3D需四面体.

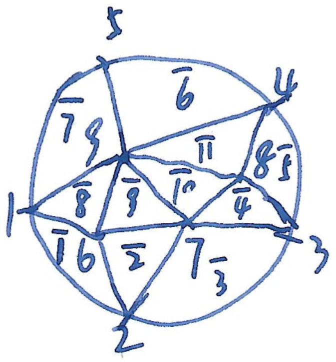
三角分割(Delauny三角分割).(后讲).



存储格式：

i：元素，三角形，按（全局）格点标号存，逆时针，从而有局域格点号

ii：全局格点标号.



格点：

$$N=\begin{pmatrix} 1 & 2 & 3 & \cdots & 9 \\ x_1 & x_2 & \vdots & & x_9 \\ y_1 & y_2 & \vdots & & y_9 \end{pmatrix}$$

三角形：

$$\begin{array}{cl} T=\begin{pmatrix} 1 & 2 & 3 & \cdots & 11 \\ 1 & 6 & 7 & & 9 \\ 2 & 2 & 2 & & 8 \\ 6 & 7 & 3 & & 4 \end{pmatrix} & \begin{array}{l} \\ \to \text{local } 1 \\ \to \text{local } 2 \\ \to \text{local } 3. \end{array} \end{array}$$

每个元素至多一条边在边界

考查1个三角形元素. 设 $\vec{r}_1, \vec{r}_2, \vec{r}_3$ 为三端点.

则有在 $\vec{r}_1$ 处为1，其余为0的基函数为

$$\phi_1(x,y) = \Big[\underbrace{\det\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}}_{=2\Delta \to \text{三角形面积}}\Big]^{-1} \cdot \det\begin{pmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}$$

类似可得 ϕ_2, ϕ_3.

$$\phi_i(x_j, y_j) = \delta_{ij}$$

对每个三角形元素均可定义3个对应的φ.

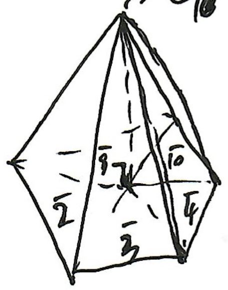
在基函数展开中:

$$\phi = \sum_j \phi_i^{T_j} \cdot C_i^{T_j}$$

其中 $C_i^{T_j}$ 只与 i 格点的全局编号有关.

以上图 7 格点为例

$\rightarrow e_7 = \phi_7(\vec{r} = (x_7, y_7))$



在 2, 3, 4, 10, 9 这些个三角形中 以7格点非0的三角形函数 系数相同.

类比1D:

解法(方程): 可遍历所有(其)格点的三角形函数, 类似于1D.

但更系统的是遍历所有三角形元素.

取元素 T_i, 其三格点全局编号为 i_1, i_2, i_3.

则 T_i 对 φ 的贡献为

$$\phi^{T_i} = C_{i_1}\phi_{i_1}^{T_i} + C_{i_2}\phi_{i_2}^{T_i} + C_{i_3}\phi_{i_3}^{T_i}.$$

试探函数: $w^{T_i} = d_{i_1}\phi_{i_1}^{T_i} + d_{i_2}\phi_{i_2}^{T_i} + d_{i_3}\phi_{i_3}^{T_i}$

$\langle w \mid \hat{L}\phi - f \rangle = 0$ 对任意 w 成立.

故 d_{i_0} 前的系数为0

方程数 = 变量数.

以 $\nabla^2\phi = -\rho$ 为例.

$$\langle w \mid \nabla^2\phi + \rho \rangle = 0$$

$$\Rightarrow \int w \nabla^2\phi \, ds = -\int w\rho \, ds$$

$$\Rightarrow -\int \nabla w \cdot \nabla\phi \, ds = -\int w\rho \, ds$$

可建立方程.

也可算 $\mathcal{L}(\varphi) = \int \left(\frac{1}{2}(\nabla\varphi)^2 - \rho\varphi\right) ds$

$$\varphi|_\Gamma = \varphi_0$$

$$\mathcal{L}(\varphi) = \sum_{T_1}^{T_N} \int_{T_i} \left(\underbrace{\tfrac{1}{2}(\nabla\varphi)^2}_{\mathcal{L}_1^i} - \underbrace{\rho\varphi}_{\mathcal{L}_2^i}\right) ds$$

$$= \sum_i \left(\mathcal{L}_1^i - \mathcal{L}_2^i\right)$$

$$= \mathcal{L}_1 - \mathcal{L}_2$$

$$C_{T_i} = \begin{pmatrix} c_{i_1} \\ c_{i_2} \\ c_{i_3} \end{pmatrix}, \quad \phi_{T_i} = \begin{pmatrix} \phi^{T_i}_{i_1} \\ \phi^{T_i}_{i_2} \\ \phi^{T_i}_{i_3} \end{pmatrix}$$

$$\phi = C_{T_i}^T \phi_{T_i}$$

$$\frac{\partial\phi}{\partial x} = C_{T_i}^T \frac{\partial\phi_{T_i}}{\partial x} = \frac{1}{2\Delta} C_{T_i}^T \begin{pmatrix} b_{i_1} \to y_{i_2} - y_{i_3} \\ b_{i_2} \to y_{i_3} - y_{i_1} \\ b_{i_3} \to y_{i_1} - y_{i_2} \end{pmatrix}$$

$$\frac{\partial\phi}{\partial y} = C_{T_i}^T \frac{\partial\phi_{T_i}}{\partial y} = \frac{1}{2\Delta} C_{T_i}^T \begin{pmatrix} d_{i_1} \to x_{i_3} - x_{i_2} \\ d_{i_2} \to x_{i_1} - x_{i_3} \\ d_{i_3} \to x_{i_2} - x_{i_1} \end{pmatrix}$$

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right)$$

$$\mathcal{L}_1^i = \frac{1}{2}\int_{T_i} \nabla\varphi\cdot\nabla\varphi\, ds$$

$$= \frac{1}{2}\cdot\frac{1}{4\Delta_{T_i}^2}\int \left[C_{T_i}^T\cdot\begin{pmatrix} b_{i_1}\\ b_{i_2}\\ b_{i_3}\end{pmatrix}\right]^2 + \left[C_{T_i}^T\cdot\begin{pmatrix} d_{i_1}\\ d_{i_2}\\ d_{i_3}\end{pmatrix}\right]^2 ds$$

$$= \frac{1}{8\Delta_{T_i}}\cdot\left[(b_{i_1}c_{i_1}+b_{i_2}c_{i_2}+b_{i_3}c_{i_3})^2+(d_{i_1}c_{i_1}+d_{i_2}c_{i_2}+d_{i_3}c_{i_3})^2\right]$$

$$= \frac{1}{2}\cdot(c_{i_1}, c_{i_2}, c_{i_3})^T\cdot K\begin{pmatrix} c_{i_1}\\ c_{i_2}\\ c_{i_3}\end{pmatrix}$$

$$K^{T_i} = \frac{1}{4\Delta_{T_i}}\begin{pmatrix} b_{i_1}^2+d_{i_1}^2 & b_{i_1}b_{i_2}+d_{i_1}d_{i_2} & b_{i_1}b_{i_3}+d_{i_1}d_{i_3}\\ b_{i_2}b_{i_1}+d_{i_2}d_{i_1} & b_{i_2}^2+d_{i_2}^2 & b_{i_2}b_{i_3}+d_{i_2}d_{i_3}\\ d_{i_3}d_{i_1}+b_{i_3}b_{i_1} & b_{i_3}b_{i_2}+d_{i_3}d_{i_2} & b_{i_3}^2+d_{i_3}^2\end{pmatrix}$$

$$= k_{ij}^{T_i}$$

$$\mathcal{L}_1 = \frac{1}{2}\sum_{T_1}^{T_n} C^{T_i\,T}\cdot K^{T_i}\cdot C^{T_i}$$

$$\mathcal{L}_2^i = C_{T_i}^T\cdot\underbrace{\int_{T_i}\rho\cdot\phi_{T_i}\,ds}_{P_{T_i}}$$

此时如何快速算积分 P_{T_i}?

1D中亦有此问题.

① ρ 认为常数, 在T_i中, $\rho=\rho_e=\frac{1}{3}(\rho_{i_1}+\rho_{i_2}+\rho_{i_3})$

② ρ 为线性函数: $\rho = \rho_{i_1}\phi_{i_1}^{T_i}+\rho_{i_2}\phi_{i_2}^{T_i}+\rho_{i_3}\phi_{i_3}^{T_i}$

$$= \rho^T\cdot\phi_{T_i}$$

对于①: $P_{T_i} = \rho_e \int_{T_i} \phi_{T_i} \, ds$

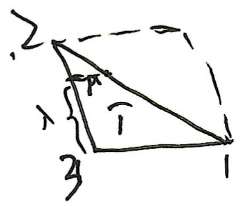
对于②: $P_{T_i} = \int \rho^{T} \cdot \phi_{T_i} \cdot \phi_{T_i} \, ds$

$$= \int \phi_{T_i}^{T} \phi_{T_i} \, ds \cdot \rho$$

需要. $\int \phi_1 \phi_2 \, ds$ 之类的积分.

算法: (1). 三角形的变换.

任意三角形 ⟶ 等腰直角三角形



补成平行四边形.

T内一点

$$\vec{r} = \lambda \vec{r_1} + \mu \vec{r_2} + \omega \vec{r_3},$$

$$\lambda + \mu + \omega = 1.$$

$$\vec{r} = \lambda(\vec{r_1} - \vec{r_3}) + \mu(\vec{r_2} - \vec{r_3}) + \vec{r_3}$$

$$0 \le \lambda \le 1, \quad 0 \le \mu \le 1, \quad \lambda + \mu \le 1$$

$\lambda = ?$ $\mu = ?$

$$\lambda(\vec{r_1} - \vec{r_3}) + \mu(\vec{r_2} - \vec{r_3}) = \vec{r} - \vec{r_3}$$

$$\lambda(\vec{r_1} - \vec{r_3}) \times (\vec{r_2} - \vec{r_3}) \cdot \hat{z} = (\vec{r} - \vec{r_3}) \times (\vec{r_2} - \vec{r_3}) \cdot \hat{z}$$

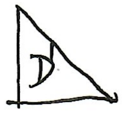
$$\Rightarrow \lambda = \frac{(\vec{r} - \vec{r_3}) \times (\vec{r_2} - \vec{r_3}) \cdot \hat{z}}{(\vec{r_1} - \vec{r_3}) \times (\vec{r_2} - \vec{r_3}) \cdot \hat{z}}$$

$$= \left[\det\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}\right]^{-1} \cdot \det\begin{pmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}$$

$$= \phi_1$$

$\mu = \phi_2$, $w = \phi_3$.

三角形 $0 \le \phi_1 \le 1$, $0 \le \phi_2 \le 1$, $\phi_1 + \phi_2 \le 1$. $\to D'$



$$\int_{T_i} ds = \int_{D'} J\,d\lambda\,d\mu = \int_{D'} J\,d\phi_1 d\phi_2.$$

$$J^{-1} = \begin{vmatrix} \frac{\partial \phi_1}{\partial x} & \frac{\partial \phi_2}{\partial x} \\ \frac{\partial \phi_1}{\partial y} & \frac{\partial \phi_2}{\partial y} \end{vmatrix}$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial \lambda} & \frac{\partial x}{\partial \mu} \\ \frac{\partial y}{\partial \lambda} & \frac{\partial y}{\partial \mu} \end{vmatrix} = \begin{vmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{vmatrix}$$

$$= (x_1 - x_3)(y_2 - y_3) - (x_2 - x_3)(y_2 - y_3)$$

$$= \vec{r_{31}} \times \vec{r_{32}} \cdot \vec{z}$$

$$= 2\Delta$$

$$\int_{T_i} \phi_{T_i}\, ds = 2\Delta \int \phi_1 \cdot d\phi_1 d\phi_2$$

$$= 2\Delta \int_0^1 \phi_1\, d\phi_1 \int_0^{1-\phi_1} d\phi_2$$

$$= 2\Delta \int_0^1 \phi_1 (1 - \phi_1)\, d\phi_1$$

$$= 2\Delta \left(\tfrac{1}{2}\phi_1^2 - \tfrac{1}{3}\phi_1^3 \right)\Big|_0^1$$

$$= \frac{\Delta}{3}$$

$$\int \phi_1^2\, ds = 2\Delta \int_0^1 \phi_1^2 (1 - \phi_1)\, d\phi_1$$

$$= 2\Delta \left(\tfrac{1}{3}\phi_1^3 - \tfrac{1}{4}\phi_1^4 \right)\Big|_0^1$$

$$= \frac{\Delta}{6}$$

$$\int \phi_1 \phi_2\, ds = 2\Delta \int_0^1 \phi_1 \int_0^{1-\phi_1} \phi_2\, d\phi_2\, d\phi_1 = \Delta \int_0^1 \phi_1 (1 - \phi_1)^2\, d\phi_1$$

$$= \Delta \left(\tfrac{1}{2}\phi_1^2 - \tfrac{2}{3}\phi_1^3 + \tfrac{1}{4}\phi_1^4 \right)\Big|_0^1$$

$$= \frac{\Delta}{12}.$$

汇总：

$$\mathcal{L} = \frac{1}{2}\sum (C^{T_i})^T k^{T_i} C^{T_i} - C_{T_i}{}^T \cdot P_{T_i}$$

$$K \cdot C = P$$

K. 对称：每个小三角形部分都对称。

正定：〃　〃（这是由定义式能看出的）。

K. 稀疏：$k_{ij} = \sum_e k^{Te}_{ij}$

遍历每个以 i 为节点的三角形元素。

$k_{ij} \neq 0$，若 ij 在某个三角形元素中。

KC=P 可用迭代法求解。

边界处理。

$$\mathcal{L} = \frac{1}{2}\sum C^T \cdot k C - C^T \cdot P$$

$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad C_1 = \begin{pmatrix} C_1^1 \\ \vdots \\ C_1^n \end{pmatrix} = \begin{pmatrix} C^1 \\ \vdots \\ C^n \end{pmatrix} \to \text{内部}$$

$$C_2 = \begin{pmatrix} C_2^1 \\ \vdots \\ C_2^m \end{pmatrix} = \begin{pmatrix} C^{n+1} \\ \vdots \\ C^{n+m} \end{pmatrix} \to \text{边界}$$

在边界上，若为第一类：$C^{n+i} = \phi(n+i)$ 为给定。

不可做变分，变分针对 C_1

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \delta C_1} = K_{11} C_1 + K_{12} C_2 - P_1 = 0$$

$$\Rightarrow K_{11} C_1 = P_1 - K_{12} C_2 = P_1'$$ （边界设为常数项）与差分法类似。

误差估计:

原则:估计局域误差、在小区域提高精度.细划网.

这样代价最小.

方法较多 需结合实际.

示例:误差来源: $L\phi=f$

ϕ 与 ϕ' 均应连续.

做三角剖分与线性近似后.ϕ 连续 ϕ' 不连续.

在某结点的真实 ϕ_i' 应为相邻元素的 ϕ' 的组合.

$$\phi_i' = \frac{1}{\sum w_{elem}} \sum_{elem} w_{elem} \phi'_{elem}$$

误差:ϕ_i' 与计算值的偏离.

如何选权重 w_{elem}?

$$\Delta = \int_D (\phi' - \phi'_{FEM})(\phi' - \phi'_{FEM})\, dS$$

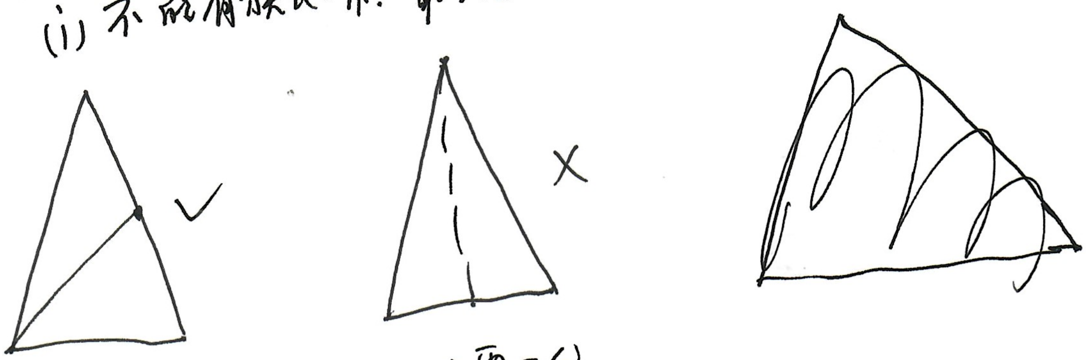
ϕ' 做线性近似:$\phi' = \sum c_i \phi_i$

变分得 c_i 的方程.

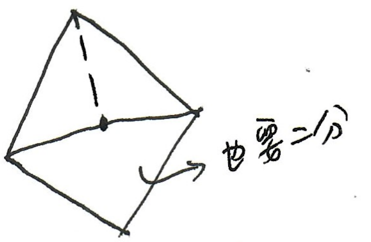
局域修正.

将三角形二分:

(i) 不能有狭长三角: 最长边分割.



(ii) 相邻的共边三角形也要二分.



算法: 若三角形T需细分, 细分T过程:

i) 找出最长边E

ii) 找出与T共享E的三角形T'

iii) 若E不是T'的最长边,
细分T'

iv) 否则, 将E的中点加入网格点集合.

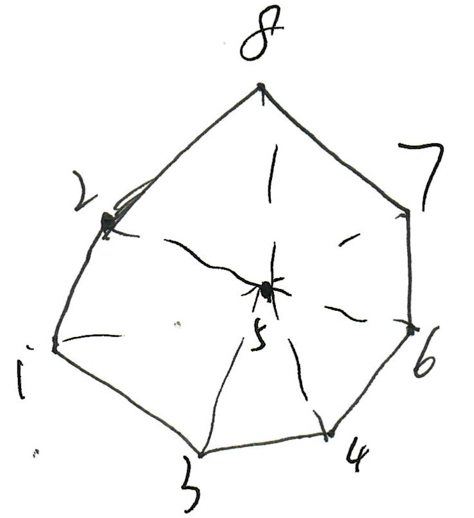
v) 根据新的网格点集合获得三角剖分.

vi) 将其改为Delaunay剖分.

注1: 对点集存储有要求, 对于每个网格点, 用一个向量按特定顺序存储其相邻网格点.

注2: 按照此点与二邻点相组成构造三角形.
为避免重复, 仅在此点标号最小时才构造.

例:



$\triangle 578$. ✓

$\triangle 125$ 在考剖时才构造