1. 考虑如下四阶 Runge-Kutta 公式

对于方程 $\dot{x}=f(x,t)$**，**

$$x_{n+1}=x_n+\frac{h}{6}(k_1+2k_2+2k_3+k_4)$$

$$k_1=f(x_n,t_n)$$
$$k_2=f\left(x_n+\frac{h}{2}k_1,t_n+\frac{h}{2}\right)$$
$$k_3=f\left(x_n+\frac{h}{2}k_2,t_n+\frac{h}{2}\right)$$
$$k_4=f(x_n+hk_3,t_n+h)$$

其中，$t_n=nh$**，**$x_n=x(t_n)$

请证明此公式确实可给出准确至四阶的解。

参考解答：先将$x(t_{n+1})$展开到四阶

$$x(t_{n+1})=x_n+hx_n'+\frac{h^2}{2}x_n''+\frac{h^3}{6}x_n'''+\frac{h^4}{24}x_n''''+\mathcal{O}(h^5)$$

要证明 Runge-Kutta 四阶公式，就要证明$x(t_{n+1})-x_{n+1}=\mathcal{O}(h^5)$。我们先算每一阶的导数

$$x_n'=f(x_n,t_n)$$

$$x_n''=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x}x_n'$$

$$x_n'''=\frac{\partial^2 f}{\partial t^2}+2f\frac{\partial^2 f}{\partial t\partial x}+f^2\frac{\partial^2 f}{\partial x^2}+\frac{\partial f}{\partial t}\frac{\partial f}{\partial x}+f\left(\frac{\partial f}{\partial x}\right)^2$$

$$x_n''''=\frac{\partial^3 f}{\partial t^3}+3f\frac{\partial^3 f}{\partial t^2\partial x}+3f^2\frac{\partial^3 f}{\partial t\partial x^2}+f^3\frac{\partial^3 f}{\partial x^3}+3f\frac{\partial f}{\partial t}\frac{\partial^2 f}{\partial x^2}$$
$$+4f^2\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x^2}+5f\frac{\partial^2 f}{\partial t\partial x}\frac{\partial f}{\partial x}+3\frac{\partial^2 f}{\partial t\partial x}\frac{\partial f}{\partial t}+\frac{\partial^2 f}{\partial t^2}\frac{\partial f}{\partial x}+\frac{\partial f}{\partial t}\left(\frac{\partial f}{\partial x}\right)^2+f\left(\frac{\partial f}{\partial x}\right)^3$$

有了这些作为参考，我们再看展开式中的阶数。由于k_1、k_2、k_3、k_4前有$\frac{h}{6}$的系数，因此我们只需要求它们展开式中的 3 阶项，因此分别将k_2、k_3、k_4做展开

$$k_4=f(t_n,x_n)+h\frac{\partial f}{\partial t}+hk_3\frac{\partial f}{\partial x}+\frac{h^2}{2}\frac{\partial^2 f}{\partial t^2}+\frac{h^2k_3^2}{2}\frac{\partial^2 f}{\partial x^2}+h^2k_3\frac{\partial^2 f}{\partial t\partial x}$$
$$+\frac{h^3}{6}\frac{\partial^3 f}{\partial t^3}+\frac{h^3k_3}{2}\frac{\partial^3 f}{\partial t^2\partial x}+\frac{h^3k_3^2}{2}\frac{\partial^3 f}{\partial t\partial x^2}+\frac{h^3k_3^3}{6}\frac{\partial^3 f}{\partial x^3}+\mathcal{O}(h^4)$$

$$k_3 = f(t_n, x_n) + \frac{h}{2}\frac{\partial f}{\partial t} + \frac{hk_2}{2}\frac{\partial f}{\partial x} + \frac{h^2}{8}\frac{\partial^2 f}{\partial t^2} + \frac{h^2 k_2^2}{8}\frac{\partial^2 f}{\partial x^2} + \frac{h^2 k_2}{4}\frac{\partial^2 f}{\partial t \partial x}$$

$$+ \frac{h^3}{48}\frac{\partial^3 f}{\partial t^3} + \frac{h^3 k_2}{16}\frac{\partial^3 f}{\partial t^2 \partial x} + \frac{h^3 k_2^2}{16}\frac{\partial^3 f}{\partial t \partial x^2} + \frac{h^3 k_2^3}{48}\frac{\partial^3 f}{\partial x^3} + \mathcal{O}(h^4)$$

$$k_2 = f(t_n, x_n) + \frac{h}{2}\frac{\partial f}{\partial t} + \frac{hk_1}{2}\frac{\partial f}{\partial x} + \frac{h^2}{8}\frac{\partial^2 f}{\partial t^2} + \frac{h^2 k_1^2}{8}\frac{\partial^2 f}{\partial x^2} + \frac{h^2 k_1}{4}\frac{\partial^2 f}{\partial t \partial x}$$

$$+ \frac{h^3}{48}\frac{\partial^3 f}{\partial t^3} + \frac{h^3 k_1}{16}\frac{\partial^3 f}{\partial t^2 \partial x} + \frac{h^3 k_1^2}{16}\frac{\partial^3 f}{\partial t \partial x^2} + \frac{h^3 k_1^3}{48}\frac{\partial^3 f}{\partial x^3} + \mathcal{O}(h^4)$$

1 阶项：

$$\frac{h}{6}[f(t_n, x_n) + 2f(t_n, x_n) + 2f(t_n, x_n) + f(t_n, x_n)] = hf(t_n, x_n) = hx_n'$$

2 阶项：

$$\frac{h}{6}\left[h\frac{\partial f}{\partial t} + hf\frac{\partial f}{\partial x} + 2\left(\frac{h}{2}\frac{\partial f}{\partial t} + \frac{h}{2}f\frac{\partial f}{\partial x}\right) + 2\left(\frac{h}{2}\frac{\partial f}{\partial t} + \frac{h}{2}f\frac{\partial f}{\partial x}\right)\right] = \frac{h^2}{2}\left(\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial x}\right) = \frac{h^2}{2}x_n''$$

3 阶项：

$$\frac{h}{6}\left[\frac{h^2}{2}\frac{\partial^2 f}{\partial t^2} + \frac{h^2 f^2}{2}\frac{\partial^2 f}{\partial x^2} + h^2 f\frac{\partial^2 f}{\partial t \partial x} + h\frac{\partial f}{\partial x}\left(\frac{h}{2}\frac{\partial f}{\partial t} + \frac{hf}{2}\frac{\partial f}{\partial x}\right)\right]$$

$$+ \frac{h}{6}\left[2\left(\frac{h^2}{8}\frac{\partial^2 f}{\partial t^2} + \frac{h^2}{8}f^2\frac{\partial^2 f}{\partial x^2} + \frac{h^2}{4}f\frac{\partial^2 f}{\partial t \partial x} + \frac{h}{2}\frac{\partial f}{\partial x}\left(\frac{h}{2}\frac{\partial f}{\partial t} + \frac{hf}{2}\frac{\partial f}{\partial x}\right)\right)\right]$$

$$+ \frac{h}{6}\left[2\left(\frac{h^2}{8}\frac{\partial^2 f}{\partial t^2} + \frac{h^2}{8}f^2\frac{\partial^2 f}{\partial x^2} + \frac{h^2}{4}f\frac{\partial^2 f}{\partial t \partial x}\right)\right]$$

$$= \frac{h^3}{6}\left[\frac{\partial^2 f}{\partial t^2} + f^2\frac{\partial^2 f}{\partial x^2} + 2f\frac{\partial^2 f}{\partial t \partial x} + \frac{\partial f}{\partial t}\frac{\partial f}{\partial x} + f\left(\frac{\partial f}{\partial x}\right)^2\right] = \frac{h^3}{6}x_n'''$$

4 阶项：这个部分比较复杂，我们分开算，先算k_2、k_3、k_4中本身带有h^3的项

$$\frac{h}{6}\left[\frac{h^3}{6}\frac{\partial^3 f}{\partial t^3} + \frac{h^3 f}{2}\frac{\partial^3 f}{\partial t^2 \partial x} + \frac{h^3 f^2}{2}\frac{\partial^3 f}{\partial t \partial x^2} + \frac{h^3}{6}\frac{\partial^3 f}{\partial x^3}\right.$$

$$+ 2\left(\frac{h^3}{48}\frac{\partial^3 f}{\partial t^3} + \frac{h^3 f}{16}\frac{\partial^3 f}{\partial t^2 \partial x} + \frac{h^3 f^2}{16}\frac{\partial^3 f}{\partial t \partial x^2} + \frac{h^3}{48}\frac{\partial^3 f}{\partial x^3}\right)$$

$$\left.+ 2\left(\frac{h^3}{48}\frac{\partial^3 f}{\partial t^3} + \frac{h^3 f}{16}\frac{\partial^3 f}{\partial t^2 \partial x} + \frac{h^3 f^2}{16}\frac{\partial^3 f}{\partial t \partial x^2} + \frac{h^3}{48}\frac{\partial^3 f}{\partial x^3}\right)\right]$$

$$= \frac{h^4}{24}\left(\frac{\partial^3 f}{\partial t^3} + 3f\frac{\partial^3 f}{\partial t^2 \partial x} + 3f^2\frac{\partial^3 f}{\partial t \partial x^2} + f^3\frac{\partial^3 f}{\partial x^3}\right)$$

在k_4中有可能出现h^3的部分为$hk_3\frac{\partial f}{\partial x} + h^2 k_3\frac{\partial^2 f}{\partial t \partial x} + \frac{h^2 k_3^2}{2}\frac{\partial^2 f}{\partial x^2}$，需要分别对应$k_3$的 3 阶、2 阶、1 阶，注意需要迭代到$k_2$中的阶数。此贡献为

$$h\frac{\partial f}{\partial x}\left[\frac{h^2}{8}\frac{\partial^2 f}{\partial t^2} + \frac{h^2}{8}f^2\frac{\partial^2 f}{\partial x^2} + \frac{h^2}{4}f\frac{\partial^2 f}{\partial t \partial x} + \frac{h}{2}\frac{\partial f}{\partial x}\left(\frac{h}{2}\frac{\partial f}{\partial t} + \frac{hf}{2}\frac{\partial f}{\partial x}\right)\right]$$

$$+ h^2\frac{\partial^2 f}{\partial t \partial x}\left(\frac{h}{2}\frac{\partial f}{\partial t} + \frac{hf}{2}\frac{\partial f}{\partial x}\right) + \frac{h^2}{2}f\frac{\partial^2 f}{\partial x^2}\left[2\left(\frac{h}{2}\frac{\partial f}{\partial t} + \frac{hf}{2}\frac{\partial f}{\partial x}\right)\right]$$

同理可得k_3中可能出现h^3的部分

$$\frac{h}{2}\frac{\partial f}{\partial x}\left(\frac{h^2}{8}\frac{\partial^2 f}{\partial t^2}+\frac{h^2}{8}f^2\frac{\partial^2 f}{\partial x^2}+\frac{h^2}{4}f\frac{\partial^2 f}{\partial t\partial x}\right)+\frac{h^2}{4}\frac{\partial^2 f}{\partial t\partial x}\left(\frac{h}{2}\frac{\partial f}{\partial t}+\frac{hf}{2}\frac{\partial f}{\partial x}\right)+\frac{h^2 f}{8}\frac{\partial^2 f}{\partial x^2}\left(\frac{h}{2}\frac{\partial f}{\partial t}+\frac{hf}{2}\frac{\partial f}{\partial x}\right)$$

由于k_2中没有这部分共吸纳，因此我们直接根据$\dfrac{h}{6}(k_1+2k_2+2k_3+k_4)$相加

$$\begin{aligned}
&\frac{h^4}{6}\left[\frac{1}{4}\frac{\partial^3 f}{\partial t^3}+\frac{3}{4}f\frac{\partial^3 f}{\partial t^2\partial x}+\frac{3}{4}f^2\frac{\partial^3 f}{\partial t\partial x^2}+\frac{1}{4}f^3\frac{\partial^3 f}{\partial x^3}+\frac{1}{8}\frac{\partial^2 f}{\partial t^2}\frac{\partial f}{\partial x}+\frac{1}{8}f^2\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x^2}+\frac{1}{4}f\frac{\partial^2 f}{\partial t\partial x}\frac{\partial f}{\partial x}\right.\\
&+\frac{1}{4}\frac{\partial f}{\partial t}\left(\frac{\partial f}{\partial x}\right)^2+\frac{1}{4}f\left(\frac{\partial f}{\partial x}\right)^3+\frac{1}{8}\frac{\partial^2 f}{\partial t^2}\frac{\partial f}{\partial x}+\frac{1}{8}f^2\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x^2}+\frac{1}{4}f\frac{\partial^2 f}{\partial t\partial x}\frac{\partial f}{\partial x}+\frac{1}{4}\frac{\partial^2 f}{\partial t\partial x}\frac{\partial f}{\partial t}+\frac{1}{4}f\frac{\partial^2 f}{\partial t\partial x}\frac{\partial f}{\partial x}\\
&\left.+\frac{1}{2}\frac{\partial^2 f}{\partial t\partial x}\frac{\partial f}{\partial t}+\frac{1}{2}f\frac{\partial^2 f}{\partial t\partial x}\frac{\partial f}{\partial x}+\frac{1}{2}f\frac{\partial f}{\partial t}\frac{\partial^2 f}{\partial x^2}+\frac{1}{2}f^2\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x^2}+\frac{f}{4}\frac{\partial f}{\partial t}\frac{\partial^2 f}{\partial x^2}+\frac{f^2}{4}\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x^2}\right]\\
&=\frac{h^4}{24}\left[\frac{\partial^3 f}{\partial t^3}+3f\frac{\partial^3 f}{\partial t^2\partial x}+3f^2\frac{\partial^3 f}{\partial t\partial x^2}+f^3\frac{\partial^3 f}{\partial x^3}+5f\frac{\partial^2 f}{\partial t\partial x}\frac{\partial f}{\partial x}+3\frac{\partial f}{\partial t}\frac{\partial^2 f}{\partial x^2}+4f^2\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial x^2}\right.\\
&\left.+\frac{\partial f}{\partial t}\left(\frac{\partial f}{\partial x}\right)^2+f\left(\frac{\partial f}{\partial x}\right)^3+3\frac{\partial^2 f}{\partial t\partial x}\frac{\partial f}{\partial t}+\frac{\partial^2 f}{\partial t^2}\frac{\partial f}{\partial x}\right]=\frac{h^4}{24}x_n^{''''}
\end{aligned}$$

证毕。