1. 考察如下一维偏微分方程：

$$y''+4x^2y=6x\cos(x^2)$$

其边条件为$y(0)=0$，$y(1)=\sin 1$。将[0,1]区间分为八份，其中间插入的七个点为$\frac{1}{4}$，$\frac{1}{2}$，$\frac{5}{8}$，$\frac{3}{4}$，$\frac{13}{16}$，$\frac{7}{8}$，$\frac{15}{16}$。在每个区间使用课上所讲的线性函数。其待求解的矩阵方程为$\boldsymbol{K\Phi}=\boldsymbol{P}$。给出矩阵$\boldsymbol{K}$与向量$\boldsymbol{P}$，求解$\boldsymbol{\Phi}$，并画出$\boldsymbol{\Phi}$随位置的变化图。

观察该微分方程得：

$$q(x)=4x^2,\quad f(x)=6x\cos(x^2)$$

定义基函数：

$$\phi_j=\begin{cases}\dfrac{x-x_{j-1}}{x_j-x_{j-1}}, & x_{j-1}<x<x_j\\ \dfrac{x_{j+1}-x}{x_{j+1}-x_j}, & x_j<x<x_{j+1}\\ 0, & \text{otherwise}\end{cases}$$

定义h：

$$h_j=x_j-x_{j-1}$$

将y用基函数展开

$$y=\sum_{j=0}^{N}c_j\phi_j$$

由于边界条件，$c_0=0$，$c_N=\sin 1\,(N=8)$。考虑如下方程，这里$i,j=1,\cdots,7$

$$\int_0^1\phi_i(y''+4x^2y-6x\cos(x^2))=0$$

将基函数展开代入并分部积分

$$I=\underbrace{-\int_0^1\phi_i'\sum_j c_j\phi_j'\mathrm{d}x}_{I_1}+\underbrace{\int_0^1\phi_i\cdot 4x^2\sum_j c_j\phi_j\mathrm{d}x}_{I_2}\underbrace{-\int_0^1\phi_i\cdot 6x\cos(x^2)\mathrm{d}x}_{I_3}$$

分别计算三部分的积分

$$I_1=-\left[\underbrace{c_i\left(\int_{x_{i-1}}^{x_i}\frac{1}{h_i}\cdot\frac{1}{h_i}\mathrm{d}x+\int_{x_i}^{x_{i+1}}\frac{1}{h_{i+1}}\cdot\frac{1}{h_{i+1}}\mathrm{d}x\right)}_{j=i}+\underbrace{c_{i+1}\int_{x_i}^{x_{i+1}}-\frac{1}{h_{i+1}}\cdot\frac{1}{h_{i+1}}\mathrm{d}x}_{j=i+1}+\underbrace{c_{i-1}\int_{x_{i-1}}^{x_i}\frac{1}{h_i}\cdot\left(-\frac{1}{h_{i-1}}\right)\mathrm{d}x}_{j=i-1}\right]$$

计算后得到

$$I_1=-c_i\left(\frac{1}{h_i}+\frac{1}{h_{i+1}}\right)+c_{i+1}\frac{1}{h_{i+1}}+c_{i-1}\frac{1}{h_i}$$

再考虑I_2，

$$I_2=\int_0^1\phi_i\cdot4x^2\cdot\sum_j c_j\phi_j\mathrm{d}x$$

$i=j$时

$$\begin{aligned}&c_i\left(\int_{x_{i-1}}^{x_i}\frac{x-x_{i-1}}{h_i}\cdot4x^2\cdot\frac{x-x_{i-1}}{h_i}\mathrm{d}x+\int_{x_i}^{x_{i+1}}\frac{x_{i+1}-x}{h_{i+1}}\cdot4x^2\,\frac{x_{i+1}-x}{h_{i+1}}\mathrm{d}x\right)\\&=c_i\left[\frac{2}{15}h_i(x_{i-1}^2+3x_{i-1}x_i+6x_i^2)+\frac{2}{15}h_{i+1}(6x_i^2+3x_ix_{i+1}+x_{i+1}^2)\right]\end{aligned}$$

$i=j-1$时

$$c_{i+1}\int_{x_i}^{x_{i+1}}\frac{x_{i+1}-x}{h_{i+1}}\cdot4x^2\cdot\frac{x-x_i}{h_{i+1}}\mathrm{d}x=c_{i+1}\left[\frac{1}{15}h_{i+1}(3x_i^2+4x_ix_{i+1}+3x_{i+1}^2)\right]$$

$i=j+1$时

$$c_{i-1}\int_{x_{i-1}}^{x_i}\frac{x-x_{i-1}}{h_i}\cdot4x^2\cdot\frac{x_i-x}{h_i}\mathrm{d}x=c_{i-1}\left[\frac{1}{15}h_i(3x_{i-1}^2+4x_{i-1}x_i+3x_i^2)\right]$$

I_2的贡献已经全部算出。其中当$i=7$时，$c_8=\sin1$，贡献一个常数项需要加到 P 矩阵中

$$\frac{1}{15}\sin1h_N(3x_{N-1}^2+4x_{N-1}x_N+3x_N^2)$$

下面计算 P 矩阵，即I_3的贡献。有两种方式，一种是将f做常量近似，另一种是利用基函数展开，这里两种方式都提供给大家参考：

常量近似即将f在区间$[x_{i-1},x_i]$上的值用$\dfrac{f(x_i)+f(x_{i-1})}{2}$代替

$$\begin{aligned}I_3&=\int_{x_{i-1}}^{x_i}\frac{x-x_{i-1}}{h_i}\cdot\frac{f(x_i)+f(x_{i-1})}{2}\mathrm{d}x+\int_{x_i}^{x_{i+1}}\frac{x_{i+1}-x}{h_{i+1}}\cdot\frac{f(x_i)+f(x_{i+1})}{2}\mathrm{d}x\\&=\frac{1}{4}h_i[f(x_i)+f(x_{i-1})]+\frac{1}{4}h_{i+1}[f(x_i)+f(x_{i+1})]\end{aligned}$$

基函数展开即$f=\sum_j f(x_j)\phi_j$，代入计算

$$
\begin{aligned}
I_3 = & \underbrace{\int_{x_{i-1}}^{x_i} \frac{x - x_{i-1}}{h_i} \cdot f_i \cdot \frac{x - x_{i-1}}{h_i} \mathrm{d}x + \int_{x_i}^{x_{i+1}} \frac{x_{i+1} - x}{h_{i+1}} \cdot f_i \cdot \frac{x_{i+1} - x}{h_{i+1}} \mathrm{d}x}_{i=j} \\
& + \underbrace{\int_{x_i}^{x_{i+1}} \frac{x_{i+1} - x}{h_{i+1}} \cdot f_{i+1} \cdot \frac{x - x_i}{x_{i+1} - x_i} \mathrm{d}x}_{i=j-1} + \underbrace{\int_{x_{i-1}}^{x_i} \frac{x - x_{i-1}}{h_i} \cdot f_{i-1} \cdot \frac{x_i - x}{h_i}}_{i=j+1}
\end{aligned}
$$

计算得

$$
I_3 = \frac{h_i}{3} f_i + \frac{h_{i+1}}{3} f_i + \frac{h_{i+1}}{6} f_{i+1} + \frac{h_i}{6} f_{i-1}
$$

不要忘记将边界的贡献加到$P(N-1)$上。此时我们得到的不含c_0,c_8的 K 矩阵应为7×7，

P 矩阵应为7×1，只需要通过

$$
\Phi = K^{-1} P
$$

就可以计算Φ。最后得到图像可以与解析解$y = x\sin x^2$进行对比。

下面是参考代码

MATLAB 代码：

```
%% 有限元方法
clear;
% 定义区间
x = [0, 1/4, 1/2, 5/8, 3/4, 13/16, 7/8, 15/16, 1];
N = 8;

% 初始化各矩阵
K = zeros(N-1);
P_basis = zeros(N-1, 1);
P_average = zeros(N-1, 1);
f = @(x) 6*x*cos(x^2);

% 计算 h
h = x(2:9) - x(1:8);

% K 和 P 矩阵元
for i = 1:N-1
    K(i,i) = - (1 / h(i) + 1 / h(i + 1)) ...
        + 2 / 15 * h(i) * (x(i)^2 + 3 * x(i) * x(i+1) + 6 * x(i+1)^2) ...
        + 2 / 15 * h(i+1) * (6 * x(i+1)^2 + 3 * x(i+1) * x(i+2) + x(i+2)^2);
    if i <= N-2
        K(i, i+1) = 1 / h(i+1) + 1 / 15 * h(i+1) * (3 * x(i+1)^2 + 4 *
x(i+1) * x(i+2) + 3 * x(i+2)^2);
        K(i+1, i) = K(i, i+1);
    end
```

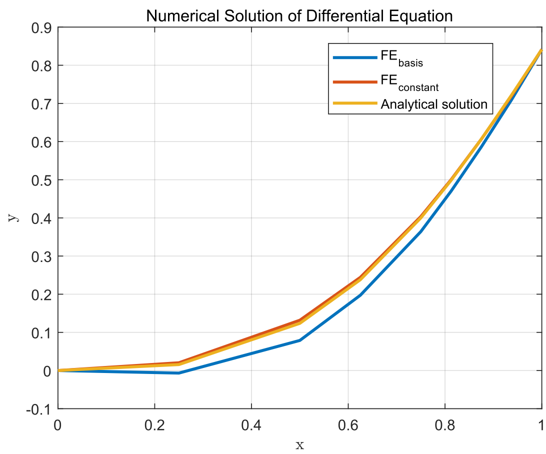
```
    P_basis(i) = h(i) / 3 * f(x(i+1)) + h(i) / 3 * f(x(i+1)) + 1/6 * h(i+1)
* f(x(i+2)) + 1 / 6 * h(i) * f(x(i));
    P_average(i) = 1/4 * h(i) * (f(x(i+1)) + f(x(i))) + 1/4 * h(i+1) *
(f(x(i+1)) + f(x(i+2)));

end
% 对 P(N-1)修正
P_basis(N-1) = P_basis(N-1) + sin(1) * (- 1/h(N) + 1/15 * h(N) * (3 *
x(N)^2 + 4 * x(N) * x(N+1) + 3 * x(N+1)^2));
P_average(N-1) = P_average(N-1) + sin(1) * (- 1/h(N) + 1/15 * h(N) * (3 *
x(N)^2 + 4 * x(N) * x(N+1) + 3 * x(N+1)^2));
% 直接取逆
Phi_basis = K \ P_basis;
Phi_basis = [0; Phi_basis; sin(1)];

Phi_average = K \ P_average;
Phi_average = [0; Phi_average; sin(1)];

plot(x, Phi_basis, LineWidth=2)
hold on;
plot(x, Phi_average, LineWidth=2)
hold on;
%% 解析解
y = x.*sin(x.^2);

% 绘图
plot(x, y, LineWidth=2);
legend('FE_{basis}','FE_{constant}','Analytical solution')
xlabel('x','Interpreter','latex');
ylabel('y','Interpreter','latex');
title('Numerical Solution of Differential Equation');
grid on;
```



Python 代码：

```
import math
import numpy as np
from matplotlib import pyplot as plt
from matplotlib import rcParams


X = [0, 1/4, 1/2, 5/8, 3/4, 13/16, 7/8, 15/16, 1]

# 定义 K 矩阵和 P 矩阵
K = np.zeros((7, 7))
P = np.zeros(7)
for i in range(1, 8):
    xi0 = X[i-1]
    xi1 = X[i]
    xi2 = X[i+1]
    hi = xi1 - xi0
    hi1 = xi2 - xi1
    fi0 = 6 * xi0 * math.cos(xi0 ** 2)
    fi1 = 6 * xi1 * math.cos(xi1 ** 2)
    fi2 = 6 * xi2 * math.cos(xi2 ** 2)
    K[i-1, i-1] = -1/hi -1/hi1 + 2/15 * hi * (6*xi1**2 + 3*xi1*xi0 + xi0**2)
    P[i-1] = 1/4 * (fi0 + fi1) * hi + 1/4 * (fi1 + fi2) * hi1
    if i > 1:
        K[i-1, i-2] = 1/hi + 1/15 * hi * (3*xi1**2 + 4*xi1*xi0 + 3*xi0**2)
    if i == 7:
        P[i-1] += - math.sin(1) * (1 / hi1 + 1 / 15 * hi1 * (3 + 4 * xi1 + 3 * xi1 ** 2))
    else:
        K[i-1, i] = 1/hi1 + 1/15 * hi1 * (3*xi2**2 + 4*xi1*xi2 + 3*xi1**2)


print(K)
Pha = np.linalg.solve(K, P)
print(Pha)
Y = [0]
y = 0
for i in range(7):
    y = Pha[i]
    Y.append(y)
Y.append(math.sin(1))

# 创建图形
plt.figure(figsize=(8, 6))
```

```
plt.plot(X, Y, marker='o', linestyle='-', color='b', label='数据线')
# 设置中文字体（以 SimHei 为例）
rcParams['font.sans-serif'] = ['SimHei']  # 设置为黑体
rcParams['axes.unicode_minus'] = False  # 解决负号显示问题
# 添加标题和标签
plt.xlabel('X 轴', fontsize=14)
plt.ylabel('Y 轴', fontsize=14)
# 添加网格和图例
plt.grid(True, linestyle='--', alpha=0.6)
plt.legend(fontsize=12)

# 显示图形
plt.show()
```